A precise analytical description of the Earth matter effect on oscillations of low energy neutrinos

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We present a formalism for the matter effects in the Earth on low energy neutrino fluxes which is both accurate and has all advantages of a full analytic treatment. The oscillation probabilities are calculated up to second order term in $\epsilon(x) \equiv 2V(x)E/\Delta m^2$ where $V(x)$ is the neutrino potential at position x. We show the absence of large undamped phases which makes the expansion in ϵ well behaved. An improved expansion is presented in terms of the variation of $V(x)$ around a suitable mean value which allows to treat energies up to those relevant for Supernova neutrinos. We discuss also the case of three-neutrino mixing.

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I. INTRODUCTION

The propagation of low energy neutrinos in the Earth [1, 2, 3] is an important aspect of physics of solar [1] - [13] and Supernova (SN) neutrinos [15] - [22]. It will be useful in determining the oscillation parameters, and, in future, to search for effects of 1-3 mixing [14] and for a 'tomography' of the Earth (see, *e.g.* [20, 21]). It might even be possible to look for small structures of the density profile [20].

In the existing calculations of Earth matter effects (see, *e.g.* [1] - [20]) the density profile is often approximated by one, two or several layers (mainly mantle and core) with constant densities or a direct numerical integration of the evolution equation is performed. However, the emergence of the large mixing MSW solution to the solar neutrino problem opens a more efficient approach to the oscillation effects in the Earth. Indeed, for the LMA parameters, the oscillations of the solar and (lower energy) supernova neutrinos inside the Earth occur in a 'weak' regime, where the matter potential V is much smaller than the 'kinetic energy' of the neutrino system, *i.e.*

$$V(x) \ll \frac{\Delta m^2}{2E}. \tag{1}$$

Here $V(x) \equiv \sqrt{2} G_F N_e(x)$, G_F is the Fermi constant, $N_e(x)$ is the number density of the electrons, $\Delta m^2 \equiv m_2^2 - m_1^2$ is the mass squared difference, and E is the neutrino energy.

In this case one can introduce a small parameter

$$\begin{aligned} \epsilon(x) &\equiv \frac{2EV(x)}{\Delta m^2} \\ &= 0.02 \cdot \left[\frac{E}{10\,\mathrm{MeV}}\right] \cdot \left[\frac{N_e(x)}{N_A}\right] \cdot \left[\frac{7.7 \cdot 10^{-5}\,\mathrm{eV}^2}{\Delta m^2}\right], \end{aligned} \tag{2}$$

where N_A is the Avogadro number, and consider an expansion of the oscillation probabilities in $\epsilon(x)$.

In ref. [23], the ϵ perturbation theory was formulated in the basis of neutrino mass states $\nu_{mass} \equiv (\nu_1, \nu_2)^T$. The oscillation probabilities and the regeneration factor were calculated to first order in ϵ. The expressions obtained are valid for arbitrary density profiles with sufficiently low density (1). They simplify the numerical calculations substantially and allow to understand in details all features of the oscillation effects. The method reproduced immediately the analytic result obtained in [24] for an approximate but realistic density profile. Similar integral expression for the regeneration factor has been discussed in [25].

Since $\epsilon(x)$ increases with energy, the lowest approximation in $\epsilon(x)$ may not be enough for larger energies. For instance, if $E \simeq 50$ MeV (possible for SN neutrinos), we find $\epsilon(x) \simeq 0.6$ at the center of the Earth.

The purpose of this paper is to improve on this method and obtain accurate formulas which are valid for higher energies. In section 2 the oscillation probabilities are calculated in second order in $\epsilon(x)$ and the convergence of the ϵ expansion is commented on. In section 3 we suggest an improved perturbation theory which allows one to extend the expansion to higher energies. The generalization to three neutrinos is given in section 4 and a brief conclusion in section 5.

II. SECOND ORDER CORRECTIONS TO THE OSCILLATION PROBABILITIES

In this and the following section we consider the mixing of two (active) neutrinos $\nu_f = U(\theta)\nu_{mass}$, where $\nu_f \equiv (\nu_e, \nu_a)^T$ and $\nu_{mass} \equiv (\nu_1, \nu_2)^T$ are the flavor and mass states, respectively and ν_a is a linear combination of ν_μ and ν_τ. $U(\theta)$ and θ are the mixing matrix and mixing angle in vacuum. We define the matrix $U(\alpha)$ as

$$U(\alpha) \equiv \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}. \tag{3}$$

In [23] the following expression for the S-matrix in the mass eigenstates basis was derived [26]:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \bar{e}^{-i\phi^m_{x_0\to x_f}} \end{pmatrix} + \\ -i\int_{x_0}^{x_f} dx \begin{pmatrix} 1 & 0 \\ 0 & \bar{e}^{-i\phi^m_{x\to x_f}} \end{pmatrix} \Upsilon(x) \begin{pmatrix} 1 & 0 \\ 0 & \bar{e}^{-i\phi^m_{x_0\to x}} \end{pmatrix} - \\ -\int_{x_0}^{x_f} dx \int_{x_0}^{x} dy \begin{pmatrix} 1 & 0 \\ 0 & \bar{e}^{-i\phi^m_{x\to x_f}} \end{pmatrix} \Upsilon(x) \begin{pmatrix} 1 & 0 \\ 0 & \bar{e}^{i\phi^m_{y\to x}} \end{pmatrix} \Upsilon(y) \begin{pmatrix} 1 & 0 \\ 0 & \bar{e}^{i\phi^m_{x_0\to y}} \end{pmatrix} \\ + \cdots , \quad (4)$$

where

$$\phi^m_{x_1\to x_2} \equiv \int_{x_1}^{x_2} dx \Delta^m(x) \quad (5)$$

is the adiabatic phase difference acquired by the neutrino eigenstates in matter on their trajectory between two points x_1 and x_2. $\Delta^m(x)$ is defined as

$$\Delta^m(x) \equiv \frac{\Delta m^2}{2E}\sqrt{1 - 2\epsilon(x)\cos 2\theta + \epsilon(x)^2}\ ; \quad (6)$$

in vacuum we obviously have

$$\Delta^m \to \Delta \equiv \frac{\Delta m^2}{2E}. \quad (7)$$

The S-matrix in (4) is written as a perturbative expansion in $\Upsilon(x)$ where

$$\Upsilon(x) = \frac{\sin 2\theta}{2}\, V(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2}\Delta^m(x)\sin^2\theta' \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (8)$$

θ' is the mixing angle of the mass eigenstates in matter,

$$\sin 2\theta' = \frac{\epsilon\ \sin 2\theta}{\sqrt{(\cos 2\theta - \epsilon)^2 + \sin^2 2\theta}} = \epsilon \sin 2\theta^m, \quad (9)$$

and $\theta^m = \theta + \theta'$ is the corresponding mixing angle of the flavor states.

The S-matrix in eq.(4) refers to a straight path through the earth from the entry point x_0 to an exit point x_f and the coordinate x is measured along the path. For notational convenience, we do not put labels x_0, x_f, *etc.* on S.

Using eq. (8), we obtain the S matrix in terms of the potential V:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \bar{e}^{-i\phi^m_{x_0\to x_f}} \end{pmatrix} - i\frac{\sin 2\theta}{2}\int_{x_0}^{x_f} dx V(x) \begin{pmatrix} 0 & \bar{e}^{-i\phi^m_{x_0\to x}} \\ \bar{e}^{-i\phi^m_{x\to x_f}} & 0 \end{pmatrix} \\ -i\frac{\sin^2 2\theta}{4\Delta} \begin{pmatrix} 1 & 0 \\ 0 & -\bar{e}^{-i\phi^m_{x_0\to x_f}} \end{pmatrix} \int_{x_0}^{x_f} dx \cdot V(x)^2 \quad (10) \\ -\frac{\sin^2 2\theta}{4}\int_{x_0}^{x_f} dx \int_{x_0}^{x} dy\ V(x)V(y) \begin{pmatrix} \bar{e}^{i\phi^m_{y\to x}} & 0 \\ 0 & \bar{e}^{-i\phi^m_{x_0\to x_f} + i\phi^m_{y\to x}} \end{pmatrix}.$$

The two last terms (proportional to ϵ^2) come from the first order in Υ (term proportional to $\sin^2\theta'$ in Eq. (8)) and the second order in Υ (see Eq. (4)) correspondingly.

Using the evolution matrix in the mass state basis (10), we can calculate the amplitudes and probabilities of various transitions. The evolution matrix from the mass states to the flavor states relevant for the solar and SN neutrinos equals US, where U is the vacuum mixing matrix (3). Consequently, the amplitude of the mass-to-flavor transition, is given by

$$A_{\nu_i\to\nu_\alpha} = U_{\alpha j}(\theta) S_{ji}. \quad (11)$$

The probability of the $\nu_2 \to \nu_e$ transition, $P_{\nu_2\to\nu_e} = |A_{\nu_2\to\nu_e}|^2 = |U_{ej}(\theta)S_{j2}|^2$ is then found to be

$$P_{\nu_2\to\nu_e} = \sin^2\theta + \frac{1}{2}\sin^2 2\theta \int_{x_0}^{x_f} dx\, V(x) \sin\phi^m_{x\to x_f} \\ + \frac{1}{4}\sin^2 2\theta\cos 2\theta \int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy V(x)\, V(y)\, \cos\phi^m_{y\to x}, \quad (12)$$

where the last term is the ϵ^2 correction. The integrations over x and y can be disentangled. Indeed, writing $\phi^m_{y\to x} = \phi^m_{y\to z} + \phi^m_{z\to x}$, where z is an arbitrary point of the trajectory, we find

$$\int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy V(x)\, V(y)\, \cos\phi^m_{y\to x} = \quad (13) \\ \left[\int_{x_0}^{x_f} dx\, V(x)\cos\phi^m_{z\to x}\right]^2 + \left[\int_{x_0}^{x_f} dx\, V(x)\sin\phi^m_{z\to x}\right]^2.$$

This shows that the second order correction is positive for all V which do not vanish.

Furthermore, for a symmetric density profile (with respect to the middle point of the trajectory) the second term in (13) vanishes. This can be seen immediately by choosing $z = \bar{x} \equiv (x_f + x_0)/2$ in the center of the trajectory. So, finally we obtain for a symmetric profile

$$P_{\nu_2\to\nu_e} = \sin^2\theta \\ + \frac{1}{2}\sin^2 2\theta \int_{x_0}^{x_f} dx\, V(x)\, \sin\phi^m_{x\to x_f} \\ + \frac{1}{4}\sin^2 2\theta\cos 2\theta \left[\int_{x_0}^{x_f} dx\, V(x)\cos\phi^m_{\bar{x}\to x}\right]^2 \quad (14)$$

or (using again the symmetry of V)

$$P_{\nu_2\to\nu_e} = \sin^2\theta \\ + \frac{1}{2}\sin^2 2\theta \sin\phi^m_{\bar{x}\to x_f} \int_{x_0}^{x_f} dx\, V(x)\, \cos\phi^m_{\bar{x}\to x} \\ + \frac{1}{4}\sin^2 2\theta\cos 2\theta \left[\int_{x_0}^{x_f} dx\, V(x)\cos\phi^m_{\bar{x}\to x}\right]^2 \quad (15)$$

The phase $\phi^m_{\bar{x}\to x_f}$ should be calculated according to (5).

The two last terms in (15) determine the regeneration parameter defined as $f_{reg} \equiv P_{\nu_2\to\nu_e} - \sin^2\theta$ (see,

e.g., [10]). The probability of the $\nu_1 \to \nu_e$ oscillations can be obtained immediately from the unitarity condition $P_{\nu_1\to\nu_e} = 1 - P_{\nu_2\to\nu_e}$.

According to (15) the effective expansion parameter of the series is

$$I \equiv \int_{\bar{x}}^{x_f} dx\, V(x) \cos\phi^m_{\bar{x}\to x}, \tag{16}$$

so that

$$P_{\nu_2\to\nu_e} = \sin^2\theta + \sin^2 2\theta \left[\sin\phi^m_{\bar{x}\to x_f} I + \cos 2\theta I^2 + ...\right]. \tag{17}$$

Notice that here the adiabatic phase should be calculated from the center of trajectory to a given point x, which corresponds to the explicit analytic expression obtained in Ref. [24]. According to (17) the first order correction is absent for trajectories with $\phi^m_{\bar{x}\to x_f} = \pi k$, ($k$ = integer) and the second order correction would be zero for maximal vacuum mixing.

Taking $\Delta_m \approx \Delta$ we obtain the useful bound

$$I \sim \frac{2E}{\Delta m^2} \int_{y(\bar{x})}^{y(x_f)} dy V(y) \cos y \leq \frac{2EV_{max}}{\Delta m^2} = \epsilon_{max}. \tag{18}$$

V_{max} is the maximum value of the potential on the trajectory and $y(x) = \frac{\Delta m^2}{2E} x$.

In eq.(10) we note the presence of a possibly large phase $\phi^m_{x_0\to x_f}$ and an undamped integral in the term $\sim V(x)^2$ (see 1-1 element of the matrix). It originates from ϵ^2 term in Υ. (The undamped terms are absent in the linear term in ϵ [27].) This could be a problem, because the potential (squared) is integrated over a large distance without an oscillatory damping, and this might give rise to a large second order term in the expansion. However by a simple partial integration of the last, $\sim V(x)V(y)$, term in (10) one can see that the undamped integral cancels. We have verified that this also happens in order V^3 for constant potentials. Therefore the ϵ expansion appears to be well behaved (see also [26]).

III. IMPROVED PERTURBATION THEORY

As mentioned before, the accuracy of our expressions decreases for higher densities and energies. However, the expansion parameter can be reduced and therefore the expansion can be improved. This can be achieved by considering a perturbation around some average potential V_0 rather than around the vacuum value $V_0 = 0$ [28]. In this case we expect the expansion parameter to be

$$\epsilon = \frac{2E\Delta V}{\Delta m^2} = \frac{2E(V - V_0)}{\Delta m^2}. \tag{19}$$

The corresponding results can be immediately obtained from the original perturbation theory. Indeed, the transition to an average potential V_0 is equivalent to considering the problem in the basis $\nu^0_m = (\nu^0_1, \nu^0_2)$, where ν^0_i are the eigenstates of the Hamiltonian in matter with a constant potential V_0. These states are analogous to mass eigenstates in the $V_0 = 0$ theory. Therefore the S-matrix S^0 for (ν^0_1, ν^0_2) follows from the S matrix for mass eigenstates (10) by the substitution

$$V \to \Delta V \equiv V - V_0, \quad \theta \to \theta^m_0, \tag{20}$$

where θ^m_0 is the flavor mixing angle in matter with the potential V_0:

$$\sin 2\theta^m_0 = \frac{\sin 2\theta}{\sqrt{1 - 2\epsilon_0 \cos 2\theta + \epsilon^2_0}} \tag{21}$$

and

$$\epsilon_0 \equiv \frac{2EV_0}{\Delta m^2}. \tag{22}$$

The adiabatic phase differences generated for the eigenstates traveling in matter with true V are invariant under a shift of the average potential, so that the phases, $\phi^m_{x_i\to x_j}$ are unchanged. Therefore

$$S^0 = S(\Delta V, \theta^m_0). \tag{23}$$

We introduce the mixing matrix

$$U'_0 \equiv U(\theta'_0) \tag{24}$$

which relates the eigenstates of neutrinos in the potential V_0 to the mass eigenstates in vacuum: $\nu_{mass} = U'_0 \nu^0_m$. The angle θ'_0 is given by

$$\sin 2\theta'_0 = \epsilon_0 \sin 2\theta^m_0 \tag{25}$$

and it is easy to check that $\theta = \theta^m_0 - \theta'_0$.

Now the amplitude of the mass-to-flavor transition, $\nu_i \to \nu_\alpha$, equals

$$A_{\nu_i\to\nu_\alpha} = U_{\alpha j}(\theta^m_0)(S^0)_{jk} U^\dagger_{ki}(\theta'_0). \tag{26}$$

A straightforward calculation leads to the $\nu_2 \to \nu_e$ oscillation probability $P_{\nu_2\to\nu_e} = |A_{\nu_2\to\nu_e}|^2$

$$P_{\nu_2\to\nu_e} = \sin^2\theta + \epsilon_0 \sin^2 2\theta_0^m \sin^2\frac{\phi^m_{x_0\to x_f}}{2} + \frac{1}{2}\sin^2 2\theta_0^m \cos 2\theta_0' \int_{x_0}^{x_f} dx\,\Delta V(x)\,\sin\phi^m_{x\to x_f}$$
$$+ \frac{\epsilon_0}{2}\sin^2 2\theta_0^m \cos 2\theta_0^m \int_{x_0}^{x_f} dx\,\Delta V(x)\,\sin\phi^m_{x_0\to x} - \frac{\epsilon_0}{8}\sin^4 2\theta_0^m \int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy \Delta V(x)\,\Delta V(y)\cos(\phi^m_{x_0\to x} - \phi^m_{y\to x_f})$$
$$+ \frac{1}{8}\sin^2 2\theta_0^m(\cos 2\theta_0^m + \cos 2\theta_0' - 2\sin^2\theta - \epsilon_0 \sin^2 2\theta_0^m)\int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy \Delta V(x)\,\Delta V(y)\cos\phi^m_{y\to x}\ . \quad (27)$$

We note that there are two first order (in ΔV) terms, one containing $\phi^m_{x_0\to x}$, the other $\phi^m_{x\to x_f}$ in contrast to the original theory which contains the phase $\phi^m_{x\to x_f}$ only.

For $V_0 = 0$ eq. (27) coincides with the previous result (12).

For a symmetric density profile we obtain

$$P_{\nu_2\to\nu_e} = \sin^2\theta + \epsilon_0 \sin^2 2\theta_0^m \sin^2\phi^m_{\bar{x}\to x_f} + \frac{1}{2}\sin^2 2\theta_0^m\,(\cos 2\theta_0' + \epsilon_0\cos 2\theta_0^m)\sin\phi^m_{\bar{x}\to x_f}\int_{x_0}^{x_f} dx\,\Delta V(x)\,\cos\phi^m_{\bar{x}\to x} +$$
$$+\frac{1}{8}\sin^2 2\theta_0^m(\cos 2\theta_0^m + \cos 2\theta_0' - 2\sin^2\theta - 2\epsilon_0\sin^2 2\theta_0^m)\left[\int_{x_0}^{x_f} dx \Delta V(x)\,\cos\phi^m_{\bar{x}\to x}\right]^2 . \quad (28)$$

Thus the effective expansion parameter of the series in the improved perturbation theory is

$$\int_{\bar{x}}^{x_f} dx\,\Delta V(x)\,\cos\phi^m_{\bar{x}\to x}. \quad (29)$$

The choice of V_0 is arbitrary; the full expansion of the S matrix does not depend on it. It just should be chosen in a clever way.

To illustrate the improvements, let us consider neutrinos with energy 50 MeV [100 MeV]. For such neutrinos ϵ=0.2 [0.4] in the upper mantle and ϵ=0.6 [1.2] in the core. Thus, the average is $\epsilon_0 \simeq 0.4$ [0.8]. Without improvement, one expects the accuracy of the computation of $\epsilon^3 \simeq 0.2$ [$O(1)$] in the core; with the improvement it is reduced to $(\epsilon - \epsilon_0)^3 \simeq 0.01$ [0.06]. The optimal V_0 can be chosen independently for each trajectory inside the Earth. For a mantle crossing trajectory, for instance, one would take the average value in the mantle.

A 'good' value of ϵ_0 may come from the observation that the second order term in (28) is multiplied by the prefactor

$$(\cos 2\theta_0^m + \cos 2\theta_0' - 2\sin^2\theta - 2\epsilon_0\sin^2 2\theta_0^m)\ . \quad (30)$$

Since ϵ_0 is arbitrary, one may choose it such that this prefactor vanishes. In Fig.1 we show the $\tan^2\theta$ dependence of ϵ_0 for which the prefactor vanishes.

In the limit $V \to 0$ the second, the third and the forth terms in (28) cancel each other (up to ϵ_0^3), and the probability reduces to $\sin^2\theta$.

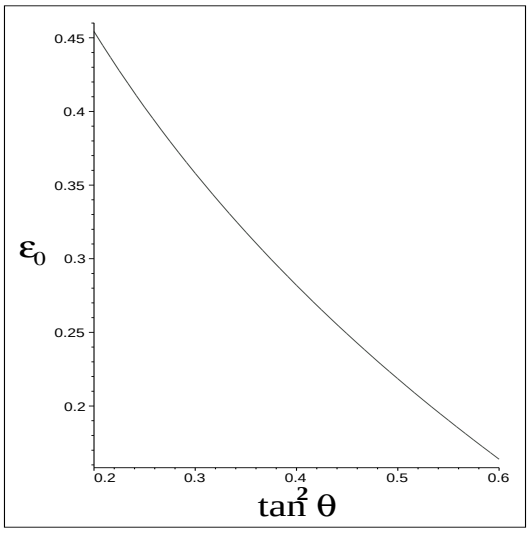

FIG. 1: The dependence of $\epsilon_0 = 2EV_0/\Delta m^2$ on $\tan^2\theta$ obtained by setting the prefactor in eq. (30) equal to zero.

IV. CORRECTIONS DUE TO THREE-NEUTRINO MIXING

In the standard parametrization the lepton mixing matrix is

$$U = O_{23}\,diag(1,1,e^{i\delta_{cp}})O_{13}\,diag(1,1,e^{-i\delta_{cp}})O_{12} =$$
$$\begin{pmatrix} c_{13}c_{12} & c_{13}s_{23} & s_{13}e^{-i\delta_{cp}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{cp}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{cp}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{cp}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{cp}} & c_{13}c_{23} \end{pmatrix}$$

By a redefinition of the mixing matrix

$$U \to U \cdot diag(1, 1, e^{i\delta_{cp}}) \tag{31}$$

the Hamiltonian becomes real, *i.e.*

$$\mathcal{H} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta_s & 0 \\ 0 & 0 & \Delta_a \end{pmatrix} + U^\dagger \begin{pmatrix} \mathrm{V} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U \tag{32}$$

$$= \begin{pmatrix} Vc_{13}^2 c_{12}^2 & Vc_{13}^2 s_{12} c_{12} & Vc_{12} c_{13} s_{13} \\ Vc_{13}^2 s_{12} c_{12} & \Delta_s + Vc_{13}^2 s_{12}^2 & Vs_{12} c_{13} s_{13} \\ Vc_{12} c_{13} s_{13} & Vs_{12} c_{13} s_{13} & \Delta_a + Vs_{13}^2 \end{pmatrix}, \tag{33}$$

where $\Delta_s \equiv \Delta m^2_\odot / 2E$ and $\Delta_a \equiv \Delta m^2_{atm}/2E - \Delta_s$.

Thus we see that both the CP phase δ_{cp} and θ_{23} do not influence the propagation in matter (determined by the Hamiltonian). Also, since in (31) the first line does not contain δ_{cp} and θ_{23} these parameters disappear in the oscillations from ν_e to ν_e, or from ν_e to mass eigenstates and vice versa. They manifest themselves only when one considers the flavor states ν_μ or ν_τ.

These arguments are general and are valid in arbitrary matter density.

We now write the Hamiltonian in the form

$$\mathcal{H} = \mathcal{H}^0_{(3\nu)} + \Upsilon_{(3\nu)}, \tag{34}$$

where

$$\mathcal{H}^0_{(3\nu)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta^m_s & 0 \\ 0 & 0 & \Delta^m_a \end{pmatrix} \tag{35}$$

and

$$\Upsilon_{(3\nu)} = \mathcal{H} - \mathcal{H}^0_{3\nu} + diag(0, \Delta_s, \Delta_a) - \frac{V + \Delta_s + \Delta_a - \Delta^m_s - \Delta^m_a}{3} I$$

$$= V\, c_{13}^2 \begin{pmatrix} 0 & \sin 2\theta_{12}/2 & c_{12} s_{13}/c_{13} \\ \sin 2\theta_{12}/2 & 0 & s_{12} s_{13}/c_{13} \\ c_{12} s_{13}/c_{13} & s_{12} s_{13}/c_{13} & 0 \end{pmatrix} + O(V^2). \tag{36}$$

Δ^m_s and Δ^m_a are the eigenvalues of the Hamiltonian in matter [29]. In eq. (36) we have subtracted a term proportional to the unit matrix in order to make it traceless and thus convenient for a power expansion.

A straightforward calculation leads to the transition probabilities of the mass eigenstates to ν_e:

$$P_{\nu_1 \to \nu_e} = c_{13}^2 c_{12}^2 - \frac{\sin^2 2\theta_{12}}{2} c_{13}^4 \int_{x_0}^{x_f} dx\, V \sin \phi_{x \to x_f} - 2\, c_{12}^2 c_{13}^2 s_{13}^2 \int_{x_0}^{x_f} dx\, V \sin \psi_{x \to x_f}, \tag{37}$$

$$P_{\nu_2 \to \nu_e} = c_{13}^2 s_{12}^2 + \frac{\sin^2 2\theta_{12}}{2} c_{13}^4 \int_{x_0}^{x_f} dx\, V \sin \phi_{x \to x_f} - 2\, s_{12}^2 c_{13}^2 s_{13}^2 \int_{x_0}^{x_f} dx\, V \sin(\psi_{x \to x_f} - \phi_{x \to x_f}), \tag{38}$$

$$P_{\nu_3 \to \nu_e} = s_{13}^2 + 2\, c_{12}^2 c_{13}^2 s_{13}^2 \int_{x_0}^{x_f} dx\, V \sin \psi_{x \to x_f} + 2\, s_{12}^2 c_{13}^2 s_{13}^2 \int_{x_0}^{x_f} dx\, V \sin(\psi_{x \to x_f} - \phi_{x \to x_f}), \tag{39}$$

where

$$\phi_{a \to b} = \int_a^b \Delta^m_s(x)\, dx\ , \qquad \psi_{a \to b} = \int_a^b \Delta^m_a(x)\, dx\ . \tag{40}$$

The function $\sin \psi_{x \to x_f}$ oscillates $\Delta^m_a / \Delta^m_s \simeq \Delta m^2_{atm} / \Delta m^2_\odot$ times faster than $\sin \phi_{x \to x_f}$. Thus, the corresponding integral is roughly $\Delta m^2_{atm} / \Delta m^2_\odot$ times smaller than the one which contains the phase ϕ; furthermore, it has a prefactor s^2_{13}. Therefore we get to a good approximation

$$P_{\nu_1 \to \nu_e} = c_{13}^2 c_{12}^2 - \frac{\sin^2 2\theta_{12}}{2} c_{13}^4 \int_{x_0}^{x_f} dx\, V \sin \phi_{x \to x_f}, \tag{41}$$

$$P_{\nu_2 \to \nu_e} = c_{13}^2 s_{12}^2 + \frac{\sin^2 2\theta_{12}}{2} c_{13}^4 \int_{x_0}^{x_f} dx\, V \sin \phi_{x \to x_f}, \tag{42}$$

$$P_{\nu_3 \to \nu_e} \simeq s_{13}^2. \tag{43}$$

These results may be also obtained from eq. (33) [25] (see [14] for some earlier discussion). If $\Delta_a \gg \Delta_s \gg V$ and $s_{13} \ll 1$, the third neutrino decouples and one arrives at the two neutrino propagation problem in matter with potential $V \to V c^2_{13}$ and mixing angle θ_{12}. Following the procedure of section II and using the full mixing matrix $U\, diag(1, 1, e^{i\delta_{cp}})$ we easily recover eqs. (41) -(43).

V. CONCLUSION

Motivated by the large mixing MSW solution to the solar neutrino we have developed a simple formulation of the earth matter effects on low energy neutrino beams. Following [23], we derive an expansion for the neutrino transitions in terms of the parameter

$$\epsilon(x) \equiv \frac{2EV(x)}{\Delta m^2}$$

to second order. By choosing a convenient constant average value for the neutrino potential as starting point, the precision can be substantially improved and it is possible to reach an accuracy of a few percents even for energies near $70 - 80$ MeV. The effective expansion parameter is a simple integral in eq. (16) (or eq. (29)) together with eq. (5)) which can be done numerically. The expansion allows for a convenient quantitative discussion of various physical effects such as the attenuation effect to the remote structures of the density profile or the effect of energy resolution of detectors. We also consider the case of three-neutrino mixing.

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[26] This result may be obtained via ordinary perturbation theory with the Hamiltonian $\mathcal{H}(x) = diag(0, \Delta) + U^\dagger diag(V(x), 0)\ U = \mathcal{H}^0 + \Upsilon$, where $\mathcal{H}^0$ is the diagonalized Hamiltonian (MSW solution) at point (x). We would like to stress that only that separation of the Hamiltonian into a non-perturbative ($\mathcal{H}^0$) and a perturbative (Υ) parts leads to results where terms proportional to the full distance traveled by neutrinos in matter are absent; this is in fact guaranteed by the existence of the MSW solution.

[27] It is important to recall that ϵ enters both through V and through the adiabatic phase ϕ^m.

[28] Even more general would be an expansion around a suitable potential for which there is a closed analytic form.

[29] When $V \ll \Delta_s \ll \Delta_a$ then $\Delta_a^m \simeq \Delta_a + O(V)$ and $\Delta_s^m \simeq \Delta_s \sqrt{(\cos 2\theta - \frac{V c_{13}^2}{\Delta_s})^2 + \sin^2 2\theta} + O(s_{13}^2 \frac{V^2}{\Delta_a})$.